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1988 J. Phys. A: Math. Gen. 21 569

(http://iopscience.iop.org/0305-4470/21/2/036)

J. Phys. A: Math. Gen. **21** (1988) 569-573. Printed in the UK

COMMENT

Random walks on fractals: higher moments†

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Received 7 July 1987

Abstract. We investigate by numerical simulation the higher moments for S_N, the number of distinct sites visited in an N-step random walk on fractal structures: the Sierpinski gasket, the Sierpinski web, and the percolation clusters of the square planar and the simple cubic lattices. We find that these moments scale similarly, e.g., the ratio of the standard deviation to $\langle S_N \rangle$, the average S_N, is constant in time.

In studying the expected values of various random-walk properties on lattices, the statistical distribution of the data is of considerable interest. Therefore, previous investigations have calculated the variance [1, 2] (square of the standard deviation σ) and several higher moments [3-6] of properties such as S_N, the number of distinct sites visited in an N-step walk. These moments are significant when testing for scaling of various properties (see, for instance, equation (6) of [6]).

It is well known that in impurity-doped lattices the expected values of the above random-walk quantities are extremely difficult to estimate analytically because of random inhomogeneities. Consequently, it is very difficult to calculate the moments of these random variables. In the present work we investigate the statistical distributions of S_N obtained from random-walk simulations on the planar 2D Sierpinski gasket, the 3D Sierpinski gasket (called the Sierpinski web), and 2D and 3D percolation clusters.

It is believed that such random-walk statistics on fractal structures are non-Gaussian [4-7]. Also, it has been conjectured that the ratio of $\langle S_N \rangle$ to its standard deviation yields a constant [4-7]. We consider the first mth moments by calculating the so-called reduced moments defined by

$$\mu_m^* = \left(\sum_{i=1}^{k} (x_i - \mu_1)^m \right) \mu_1^{-m} \tag{1}$$

where μ_1, the first moment, is the arithmetic mean:

$$\mu_1 = k^{-1} \sum_{i=1}^{k} x_i. \tag{2}$$

For Euclidean lattices the first two moments are well known [1, 2]:

† Supported by NSF Grant DMR 8303919.

0305-4470/88/020569+06$02.50

$$\langle S_N \rangle = \begin{cases} (8N/\pi)^{1/2} \\ \pi N/\ln N \\ CN \end{cases} \qquad \sigma^2_{\langle S_N \rangle} = \begin{cases} N & \text{1D} \\ N^2/\ln^4 N & \text{2D} \\ N \ln N & \text{3D.} \end{cases} \tag{3}$$

For Gaussian statistics μ_m^* is a monotonically decreasing function; for the fractal structures, the conjecture implies that μ_m^* is constant. If this is correct, it is useful not only in determining the accuracy of simulation results on fractals but also in bounding the fractal-to-Euclidean crossover regime. The onset of the crossover could then be characterised in terms of moving from non-Gaussian to Gaussian statistics, i.e. from a constant to a monotonically decreasing μ_m^*.

The simulation methods are described in greater detail in [8-11], and a brief description is included below. Results and a discussion complete this comment.

For simulations on the planar Sierpinski gasket and the Sierpinski web, single random walkers started at random initial sites; a site was defined as the vertex of a triangle. The direction of motion at each time step N was decided by a number selected from a uniform pseudorandom distribution. A walker moved with equal probability to any of its z nearest-neighbour sites, where $z = 4$ for the Sierpinski gasket and $z = 6$ for the Sierpinski web. A total of 3000 realisations were performed on the eighth-order planar Sierpinski gasket ($\simeq 10^4$ sites) with $N = 2000$ steps; in addition, 5000 realisations were performed on the sixth-order Sierpinski web ($\simeq 10^4$ sites) with $N = 2000$ steps.

Calculations on percolating clusters were performed using the recently described method [10] of generating such clusters exactly at their critical threshold (and not just on the average). Random-walk motion was followed using the myopic ant model for a total of $N = 5000$ steps, and for 10 000 realisations. The walker was initially positioned at some random site of the percolation cluster, and its direction of motion was decided by a pseudorandom variable with uniform distribution. The size of the lattices employed was 300×300, and only one random-walk event was performed on each lattice.

For figure 5 (see below) the data from [11] were used. Here the cluster-growth technique was utilised on lattices of size 157^3 and the random-walk process was followed for $N = 2 \times 10^5$ steps. This technique allowed the random-walk process to occur on clusters of any size in contrast with the previous method which restricted the random walk to the largest cluster.

The reduced moments for the planar Sierpinski gasket (figure 1) and the Sierpinski web (figure 2) are seen to be reasonably constant for μ_2^*, μ_3^* and μ_4^*. In fact, the largest fluctuation seems to be in the second reduced moment. More extensive simulations based on 10 000 realisations [4, 5] have shown that μ_2^* is indeed constant for the planar Sierpinski gasket.

Figure 3 shows the S_N distribution for random walks on the infinite percolating cluster. Since these clusters are generated exactly at criticality, it is expected that the random walks will exhibit truly fractal behaviour. Indeed, as it has been shown [10], the spectral dimension d_s in the relationship $S_N \sim N^{d_s/2}$ is $d_s = 1.30 \pm 0.02$, in excellent agreement with the generally accepted conjectures for percolating clusters. One observes in figure 3 that the S_N distribution at step $N = 5000$ is not symmetric, but nevertheless it is relatively smooth. The different moments are shown in figure 4. Here the second moment is relatively smooth and constant with respect to time.

Figure 5 shows a plot of σ/S_N for three-dimensional simple cubic lattices slightly below, around and above the critical percolation threshold. This calculation is performed using the cluster-growth technique and thus the point of origin for the random walk may be on a cluster of any size and not just the largest infinite cluster. One sees

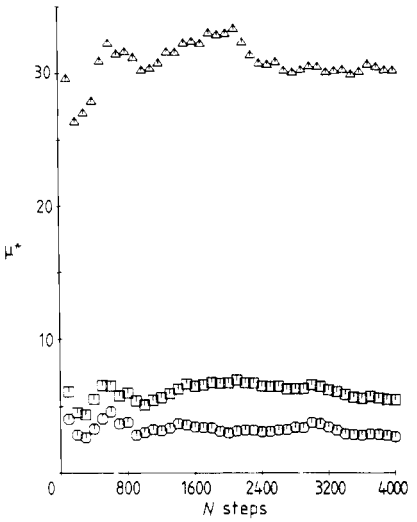

Figure 1. Reduced moments μ_2^* (triangles), μ_3^* (circles) and μ_4^* (squares) plotted against time. 5000 realisations on an eighth-order planar Sierpinski gasket followed single random walkers for 4000 steps.

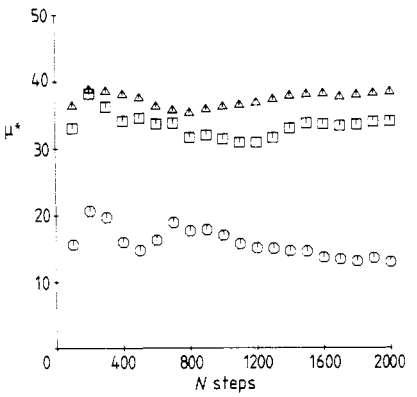

Figure 2. Reduced moments μ_2^* (triangles), μ_3^* (circles) and μ_4^* (squares) plotted against time. 3000 realisations on a sixth-order Sierpinski web followed single random walkers for 2000 steps.

that the only constant curve occurs around the critical point $p_c = 0.312$ while above this point the curves increase at different rates. This is just another manifestation of the crossover from fractal to Euclidean behaviour shown in figures 6 and 7 of [11].

In summary, the conjecture that higher moments all scale in the same fashion for fractal structures is quite good, based on the systems which we considered here in two and three dimensions: the percolation clusters at criticality and the Sierpinski structures. Not only is this conjecture useful in assessing the accuracy of simulation results on fractals, but it may also provide a means of bounding the beginning of the fractal to Euclidean crossover regime with the transition from non-Gaussian to Gaussian statistics. The final stages of the crossover regime are best characterised with the use of the formulae in (3).

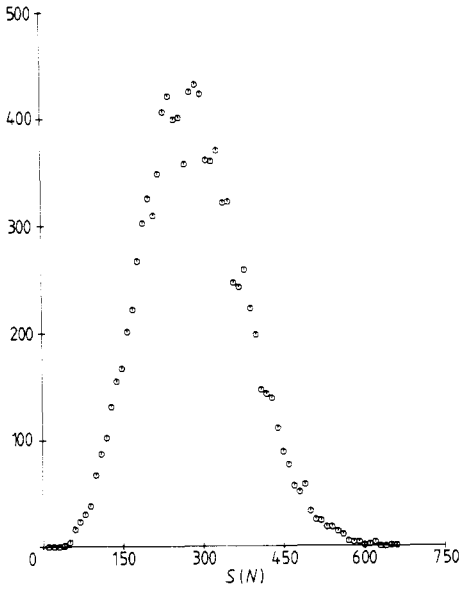

Figure 3. The S_N distribution from the random walks on the infinite percolating cluster exactly at criticality. This histogram contains 10 000 realisations (see text for details of calculations). These results are for $N = 5000$ steps, where $\langle S_N \rangle = 288.0$.

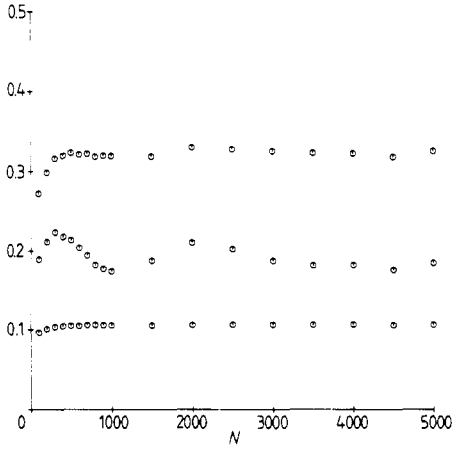

Figure 4. Several different reduced moments from random walks on the infinite percolating clusters exactly at criticality. The data are the same as in figure 3. Top to bottom: fourth, third and second moments. In order to bring all three curves onto the same scale they have been multiplied by the factors 10^9, -2×10^5 and 1, top to bottom, respectively.

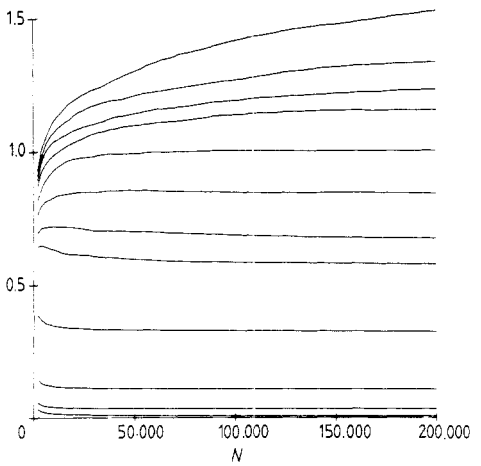

Figure 5. Plot of $\sigma/\langle S_N \rangle$ as a function of time (N is number of steps) for three-dimensional simple cubic lattices. The different lines correspond to different occupational probabilities p, as $p = 1.00$, 0.75, 0.50, 0.40, 0.35, 0.34, 0.33, 0.325, 0.32, 0.3175, 0.315, and 0.31 from top to bottom. Here the random walk process may originate on any-size cluster (and not just the largest cluster).

References

[1] Montroll E W and West B J 1979 *Fluctuation Phenomena* ed E W Montroll and J L Lebowitz (Amsterdam: North-Holland) ch 2
[2] Zumofen G and Blumen A 1982 *J. Chem. Phys.* **76** 3731
[3] Torney D C 1986 *J. Stat. Phys.* **44** 49
[4] Blumen A, Klafter J and Zumofen G 1983 *Phys. Rev.* B **28** 6112
[5] Angles d'Auriac J C, Benoit A and Rammal R 1983 *J. Phys. A: Math. Gen.* **16** 4039
[6] Zumofen G, Blumen A and Klafter J 1984 *J. Phys. A: Math. Gen.* **17** L479
[7] Rammal R 1984 *J. Stat. Phys.* **36** 547
[8] Argyrakis P, Anacker L W and Kopelman R 1984 *J. Stat. Phys.* **36** 579
[9] Anacker L W, Kopelman R and Newhouse J S 1984 *J. Stat. Phys.* **36** 591
[10] Keramiotis A, Argyrakis P and Kopelman R 1985 *Phys. Rev.* B **31** 4617
[11] Argyrakis P and Kopelman R 1985 *J. Chem. Phys.* **83** 3099